ON THE STABILITY OF SOLUTIONS OF THE NONLINEAR EQUATION OF HEAT CONDUCTION

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There exist many ways of investigating the stability of the nonlinear equation of heat conduction. The problem has been discussed in a paper by Bellman [1] in connection with a domain in the form of a parallelepiped; Mlak [2] considered a more general problem for a system of parabolic type. More references concerning this problem can be found in [1] and [2].

In the present paper we shall consider the stability of the solution of the nonlinear equation of heat conduction in a domain C_{L2}. Making use of some of the ideas contained in Liapunov's second method, we shall arrive at sufficient conditions for asymptotic stability.

Let G be a bounded domain in the space $E^m(x_1, \ldots, x_m)$. The boundary of G will be denoted by Γ. We consider the following problem for the nonlinear equation of heat conduction in domain G for $0 < t < \infty$:

$$\frac{\partial u}{\partial t} = \triangle u + f(x, t, u), \qquad u = 0 \quad \text{on } \Gamma, \qquad u = \varphi(x) \quad \text{for } t = 0 \tag{1}$$

Here ∇^2 denotes Laplace's operator in the domain G and the intensity of the sources distributed over the domain, it being postulated that $f(x, t, u) = O(u)$ when $u = 0$.

Let $M = u(x, t)$ be a class of continuous functions possessing continuous derivatives $\partial u/\partial t$ and $\partial^2 u/\partial x_i \partial x_j$. Introducing the norm [3]

$$\| u \| = \left(\int_G u^2(x, t)\, d\Omega \right)^{1/2} \tag{2}$$

we obtain a normalized space which we denote by C_{L2}. If $\phi = 0$, then the problem possesses a trivial solution $u = 0$. It is interesting to determine under what conditions it can be assumed that the solution of problem (1) tends to zero as $t \to \infty$. The process of tending to zero is understood to be according to the norm C_{L2}.

Definition. The vanishing solution of problem (1) is stable if for any positive number ϵ there exists a δ such that $\|u\| \leqslant \epsilon$ for $t > 0$ if $u = 0$ on Γ and $\|u(x, 0)\| \leqslant \delta$.

We seek a solution of problem (1) in the form

$$u(x, t) = a_1(t) z_1(x) + a_2(t) z_2(x) + \dots \tag{3}$$

where $a_i(t)$ denote provisionally undetermined coefficients which depend on time and $z_i(x)$ denote eigenfunctions of the operator $\nabla^2 z + \lambda z = 0$ in G with the boundary conditions $z = 0$ on the contour Γ. It is seen from (3) that the boundary condition is satisfied automatically. If (3) is substituted into (1) the solution of the problem reduces itself to the solution of an infinite system of nonlinear ordinary differential equations. Substituting (3) into (1) and bearing in mind the properties of the functions z_i, i.e. $\nabla^2 z_i = \lambda_i z_i$, we obtain

$$\sum_{i=1}^{\infty} \left(\frac{da_i}{dt} + \lambda_i a_i\right) z_i = f(x, t, u) \tag{4}$$

where on the right-hand side it is understood that u stands for the expression in (3).

It is possible to assume that the system of functions $\{z_i\}$ is orthonormalized.

Multiplying both sides of (4) by z_j and integrating over the domain G, we obtain

$$\frac{da_j}{dt} + \lambda_j a_j = \int_G f(x, t, u) z_j \, d\Omega \qquad (j = 1, 2, \dots) \tag{5}$$

We shall assume $a_i(t) = \exp(-\beta t) . b_i(t)$. The system (5) with the new variables $b_i(t)$ assumes the form

$$\frac{db_j(t)}{dt} = (-\lambda_j + \beta) b_j + e^{\beta t} \int_G f\left(x, t, e^{-\beta t} \sum_{i=1}^{\infty} b_i z_i\right) z_j \, d\Omega \qquad (j = 1, 2, \dots) \tag{6}$$

We shall consider the positive-definite function

$$V = \frac{1}{2} [b_1^2(t) + b_2^2(t) + \dots] \tag{7}$$

The total derivative of V with respect to t, formed on the assumption that the variables $b_i(t)$ satisfy the infinite system of differential equations (6), is

$$\frac{dV}{dt} = \sum_{j=1}^{\infty} (-\lambda_j + \beta) b_j^2 + e^{\beta t} \int_G f\left(x,\ t,\ e^{-\beta t} \sum_{i=1}^{\infty} b_i z_i\right) \left(\sum_{j=1}^{\infty} b_j z_j\right) d\Omega \tag{8}$$

We shall suppose that the function $f(x,\ t,\ u)$ satisfies the condition

$$|f(x,\ t,\ u)| \leqslant \alpha(t) \| u \|_{C_{L_2}} | u | \tag{9}$$

uniformly on $x \in G$, where $a(t)$ is an arbitrary function, but such that starting at some instant, $a(i)\ \exp(-\beta t)$ does not increase.

Making use of (9) for an arbitrary function V we obtain

$$\frac{dV}{dt} \leqslant 2(-\lambda_{\min} + \beta) V + e^{\beta t} \alpha(t) \| u \| \int_G | u | \left| \sum_{i=1}^{\infty} b_i z_i \right| d\Omega$$

Here $\lambda_{\min}$ denotes the smallest characteristic value of the operator $\nabla^2 z + \lambda z = 0$ in domain G and $z = 0$ on contour Γ. Remembering that

$$| u | = e^{-\beta t} \left| \sum_{i=1}^{\infty} b_i z_i \right|, \qquad \| u \| = \sqrt{2} e^{-\beta t} V^{1/2}$$

we obtain

$$\frac{dV}{dt} \leqslant 2(-\lambda_{\min} + \beta) V + \sqrt{2} \alpha(t) e^{-\beta t} V^{1/2} \int_G \left(\sum_{i=1}^{\infty} z_i b_i\right)^2 d\Omega \tag{10}$$

or, which amounts to the same

$$\frac{dV}{dt} \leqslant 2(-\lambda_{\min} + \beta) V + 2\sqrt{2} e^{-\beta t} \alpha(t) V^{3/2} \tag{11}$$

We shall assume that $\exp(-\beta t).a(t)$ does not increase starting from some $t = t_0$, and that at instant $t = t_0$

$$b_1^2(t_0) + b_2^2(t_0) + \ldots \leqslant \varepsilon, \qquad \sqrt{2} \alpha(t_0) e^{-\beta t_0} [V(t_0)]^{1/2} \leqslant \eta \tag{12}$$

Remembering that $V(t_0) \leqslant 1/2\ \epsilon$, and choosing ϵ so small as to satisfy the inequality $-\lambda_{\min} + \beta + \eta \leqslant 0$, we obtain $(dV/dt)_{t=t_0} \leqslant 0$, and consequently inequality (12) remains valid also for $t > t_0$.

Taking into account inequality (12), inequality (11) leads to

$$\frac{dV}{dt} \leqslant 2(-\lambda_{\min} + \beta + \eta) V, \ t > t_0$$

If $\lambda_{\min} \geqslant \beta + \eta$ then the derivative of the positive-definite function V is negative, and consequently the function V cannot increase from $t = t_0$ onwards. It follows that all solutions of the system (6) which enter domain (12) cannot leave it later. Since

$$\|u\| = \sqrt{2}\, e^{-\beta t}\, |^{1/2}, \quad \text{as } \|u\| \to 0 \text{ then } t \to \infty$$

We shall assume that in the neighborhood of the vanishing solution for a finite segment $0 \leqslant t \leqslant t (T \geqslant t_0)$ the solution is unique. Then there must exist a domain of initial values $b_1^2(0) + b_2^2(0) \ldots \leqslant \delta (\delta > 0)$ for $t = 0$, such that all solutions of system (6) which leave this domain enter the domain $b_1^2(0) + b_2^2(0) + \ldots \leqslant \epsilon$ for $t = t_0$. To every system $b_i(0)$ there corresponds an initial function $\phi(x)$ which satisfies the condition $\|\phi(x)\|^2 = b_1^2(0) + b_2^2(0) + \ldots$, where $b_i(0)$ are the Fourier coefficients of element $\phi(x)$.

The preceding result will now be formulated as a theorem.

The norm of the solution of problem (1) in domain C_{L2} tends asymptotically to zero as $t \to \infty$, if:

a) the function $f(x, t, u)$ on the right-hand side of Equation (1) satisfies the condition

$$|f(x, t, u)| \leqslant \alpha(t) \|u\|_{C_{L_2}} |u|$$

where the characteristic number [4, p. 162] of the positive function $\alpha(t)$ is larger than β;

b) the norm of the initial function $\phi(x)$

$$\|\varphi\| \leqslant \delta$$

c) the smallest positive number λ_{min} of the operator $\nabla^2 z + \lambda z = 0$ in domain G with $z = 0$ on contour Γ is

$$\lambda_{\min} \geqslant \beta + \eta$$

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